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# Exact Controllability of Backward Stochastic Control Systems

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***Abstract***

This paper gives a necessary condition of exact controllability for backward stochastic control systems, and introduces three equivalent conditions of necessary and sufficient for the linear system to be exactly controllable. Finally, as an application of this stochastic control system, two examples on the optimal portfolio problem with consumption will be given.

***Keywords***

*Backward Stochastic Control Systems; Exact Controllability; Applications*

## Introduction

In the view of backward stochastic differential equation (BSDE), Peng[1]has firstly defined the exactly terminal-controllable and exactly controllable for a forward stochastic control system, then he gave a necessary condition to the exactly terminal-controllable for the stochastic control system with deterministic random coefficients, finally he gave a necessary and sufficient condition for stochastic exact controllability and algebraic criterion for the linear system. Y Liu, Peng[2,3]discussed the exact controllability of stochastic control system when the control energy constraint was concerned, then obtained the necessary and sufficient condition which determined linear stochastic systems to be exactly controllable with control constraint by using the method of moment theory. F Liu[4]has discussed the nonlinear stochastic control systems which the coefficient is time-dependent, and gave a necessary and sufficient condition for the system to be exactly controllable. Based on the study of Peng and Liu, H Li and M Yao[5] gave a necessary and sufficient conditions for linear stochastic control system whose coefficient is time-dependent.

Along with the rapid progress of financial theory, BSDE plays a more and more important role in financial research. Z Wu and W Xu[6] have studied the problem of the European option pricing and got a European option price formula when dividend was considered and asset and portfolio were satisfied some conditions. H Wang, X Wang[7] discussed an optimal control problem on portfolio and consumption choice in international security market and they have obtained the explicit optimal portfolio and consumption rates.

In this paper, the exact controllability of backward stochastic control problem will be cinsidered. And let $(\Omega, F, P)$ be a complete probability space, $\{B_t, t \geq 0\}$ is a one-dimensional standard Wiener process in this space, $\mathsf{F}_t = \sigma\{B_s, 0 \leq s \leq t\}$ denotes its natural filtration, $U[0,T] \subset R^k$ is the set of admissible controls. All the processes in this paper are $F_t -$adapted and square integrals, and the set of all $R^m -$valued square integral processes is denoted by $L_F^2(0,T;R^m)$.

In section 2, we propose definitions of initial-controllable and exact control for backward stochastic control system; at the same time some conditions will be given for each of them. Section 3 gives three equivalent conditions for linear system to be exact control. In section 4, as applications of the linear control system, we give two examples on the optimal portfolio problem with consumption.

## Exact Controllability Backward Stochastic Control Systems

In this section, we will study the following backward stochastic control system

$$\begin{aligned} -dy_t &= f(y_t, z_t, v_t)dt - z_t dB_t \qquad t \in [0,T] \\ y_T &= \xi \end{aligned} \tag{1}$$

with initial point

$$y(0) = y_0 \in R^m \tag{2}$$

here

$$f(y,z,v): R^m \times R^m \times R^k \to R^m, \ \xi \in L_F^2(\Omega, F_T, P, R^m)$$

The definitions of exactly terminal-controllable and exactly controllable for forward stochastic control system have been given by Peng [1], on the basis of these, we will give analogous definitions about control system (1).

**Definition 2.1** A stochastic control system (1) is called initial-controllable, if for any $y_0 = y \in R^m$, there exists at least one admissible control $v_t \in U[0,T]$ and a variable $\xi \in L_F^2(\Omega, F_T, P, R^m)$ such that the corresponding trajectory $(y_t, z_t)$ satisfies the initial condition (2).

**Definition 2.2** For any $\xi \in L_F^2(0,T;R^m)$ and $y_0 \in R^m$, a stochastic control system (1) is called exact control, if there exists at least one admissible control $v_t \in U[0,T]$, such that the corresponding $(y_t, z_t)$ satisfies the initial condition(2).

Let us consider the following stochastic differential equation (SDE) firstly

$$\begin{aligned} dy_t &= f(y_t, z_t, v_t)dt + z_t dB_t \qquad t \in [0,T] \\ y(0) &= y_0 \end{aligned} \tag{3}$$

By comparing the system (1) and SDE (3), if we regard a trajectory and the related admissible control $(y_t, z_t, v_t)$ as a pair of solution processes $(y_t, z_t, v_t)$ of SDE (3) , then initial-controllability is just equivalent to the well-posed of the SDE (3). For SDE (3) we have the following existence and uniqueness theorem.

**Theorem 2.1** [8] (existence and uniqueness of solution to SDE ) For a SDE

$$\begin{aligned} dx_t &= b(t, x_t)dt + \sigma(t, x_t)dB_t \qquad t \in [0,T] \\ x(0) &= x_0 \end{aligned} \tag{4}$$

Let $b(\cdot,\cdot):[0,T]\times R^n \to R^n, \sigma(\cdot,\cdot):[0,T]\times R^n \to R^{n\times m}$ be measurable functions, and $E[|x_0|^2] < \infty$, for some constant $C > 0$ satisfying

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|) \qquad x \in R^n, t \in [0,T]$$

and some constant $D > 0$ satisfying

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le D|x-y| \qquad x, y \in R^n, t \in [0,T]$$

then SDE (4) exists unique solution.

For SDE (3), it is easy to verify that, when $z_t \in L_F^2(0,T;R^m)$ and $v_t \in U[0,T]$ are given, assuming $E[|y_0|^2] < \infty$, for some constant $C > 0$ satisfies

$$|f(t,y,z,v)| \le C(1+|y|) \qquad y \in R^m, t \in [0,T]$$

and some constant $D > 0$ satisfies

$$|f(t,x,z,v) - f(t,y,z,v)| \le D|x-y| \qquad x, y \in R^m, t \in [0,T]$$

then SDE (3) exists unique solution. It is easily for us to get the following theorem

**Theorem2.2** A necessary condition for a backward stochastic control system (1) to be initial-controllable is that for some constant $C > 0$ satisfying

$$\left| f(t,y,z,v) \right| \le C(1+|y|) \qquad y \in R^m, t \in [0,T]$$

and some constant $D > 0$ satisfying

$$\left| f(t,x,z,v) - f(t,y,z,v) \right| \le D|x-y| \qquad x,y \in R^m, t \in [0,T]$$

The proof of this theorem is obviously.

## Exact Control Criterions of Linear Backward Stochastic Control System

For this part, we will discuss exact control of the following linear stochastic control system

$$\begin{aligned} -dy_t &= (Ay_t + A_1 z_t + Bv_t)dt - z_t dB_t \qquad t \in [0,T] \\ y_T &= \xi \end{aligned} \tag{5}$$

here, $A$ and $A_1$ are $m \times m$ matrices, $B$ is $m \times k$ matrices, and $\xi \in L^2(\Omega, F_T, P, R^m)$ .For the following BSDE:

$$\begin{aligned} -dy_t &= (Ay_t + A_1 z_t + Bu_t)dt - z_t dB_t \qquad t \in [0,T] \\ y_T &= 0 \end{aligned} \tag{6}$$

Set $S = \left\{ y_0{}^u ; u \in L_F^2(0,T;R^k) \right\}$ , then we get the following theorem

**Theorem3.1** Exact control criterions of stochastic control system (5)

$(i)$ $S = \left\{ y_0{}^u ; u \in L_F^2(0,T;R^k) \right\} = R^m = R^m$ ;

$(ii)$ $Rank(B, AB, A_1B, AA_1B, A_1AB, \cdots) = m$ ;

$(iii)$ Matrix $G = E\int_0^T (x_tB)(x_tB)^T dt$ is positive.

Here $x_t$ in $(iii)$ is the solution of the following SDE, and $I_m$ is m-dimensional unit vector

$$\begin{aligned} dx_t &= x_t(Adt + A_1 dB_t) \qquad t \in [0.T] \\ x_0 &= I_m \end{aligned} \tag{7}$$

Further more, optimum control is $u_t = (x_tB)^T G^{-1} y_0$ .

**Proof** Firstly, from the definition of $S$ , it follows immediately that dissertation $(i)$ is equivalent to exact controllability of system (5).

$(i) \Rightarrow (ii)$ In Ref. [2], we know that

$$\left\{ y_0{}^u ; u \in L_F^2(0,T;R^k) \right\} = Span[B, AB, A_1B, AA_1B, A_1AB, \cdots]$$

Consequently $Rank(B, AB, A_1B, AA_1B, A_1AB, \cdots) = m$

$(ii) \Rightarrow (iii)$ Suppose that Matrix $G = E\int_0^T (x_tB)(x_tB)^T dt$ is not positive, then there exists a non-zero $\alpha \in R^m$ such that

$$\alpha^T G \alpha = E\int_0^T (\alpha^T x_t B)(\alpha^T x_t B)^T dt = 0 \quad \text{a.s.} \quad \forall t \in [0,T]$$

that is $\alpha^T x_t B = 0$ . This is impossible, so $G$ is positive.

$(iii) \Rightarrow (i)$ Using $It\hat{o}'s$ formula applied to $x_t \cdot y_t$ , then

$$y_0 = E\int_0^T x(t)Bu(t)dt \tag{8}$$

that is , for any admissible control $u_t \in U[0,T]$ will satisfy (8). Then for any given $y_0$ , since $rank(G) = m$ , we can find a control $u_t = (x_t B)^T G^{-1} y_0$ which satisfies (8). That means for any given $y_0 \in R^m$ , we can find an admissible control $u_t \in L_F^2(0,T;R^k)$ which is satisfied (8), so $S = \left\{ {y_0}^u ; u \in L_F^2(0,T;R^k) \right\} = R^m$

Next we will prove that the admissible control $u_t = (x_t B)^T G^{-1} y_0$ is optimum control.

Suppose that $u_t, u_t' \in U[0,T]$ are two different admissible control, so they all satisfy (8), then

$$0 = E\int_0^T x(t)B(u_t - u_t')dt \tag{9}$$

multiply ${y_0}^T (G^{-1})^T$ on both sides of (9) , we can get

$$E\int_0^T \|u_t\|^2 dt = E\int_0^T u_t^T u_t' dt$$

$$0 \le E\int_0^T \left\| u_t - u_t' \right\|^2 dt = E\int_0^T (\|u_t\|^2 - 2u_t^T u_t' + \left\| u_t' \right\|^2)dt = E\int_0^T (\|u_t'\|^2 - \|u_t\|^2)dt$$

that is

$$E\int_0^T \|u_t'\|^2 dt \ge E\int_0^T \|u_t\|^2 dt$$

then $u_t = (x_t B)^T G^{-1} y_0$ is the optimum control.

## Applications

BSDEs and optimal control theory were effective tools in the research of the portfolio problem. In the following this paper gives two examples of the backward stochastic control system. Different from the general portfolio problem, we consider the portfolio problem with consumption.

**Example 4.1** Consider a simple stock market, there is only one kind of stocks and a bond, assume that stock market is complete and transaction cost is not in the considered range. The bond pricing process satisfies

$$dp_0(t) = rp_0(t)dt \tag{10}$$

$p(t)$ is the pricing process of the stock

$$dp(t) = \alpha p(t)dt + \sigma p(t)dB_t \tag{11}$$

The investment value at moment $t$ of bond is

$$dS_0(t) = [rS_0(t) - C(t)]dt \tag{12}$$

and the stock is

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB_t \tag{13}$$

where $C(t)$ is consumer flow at moment $t$ .

Let $y(t) = S_0(t) + S(t)$ is the total investments of investors at moment $t$ , and assume that $y(T) = \xi$ , then from (10) to (13) we can obtain

$$\begin{aligned} &dy(t) = [ry(t) + (\alpha - r)S(t) - C(t)]dt + \sigma S(t)dB_t \\ &y(T) = \xi \end{aligned} \tag{14}$$

As $\sigma \in R$ is a constant, if we let $\sigma S(t) = z(t)$ , then (14) can be rewritten

$$\begin{aligned} &-dy(t) = [-ry(t) + \frac{r-\alpha}{\sigma} z(t) + C(t)]dt - z(t)dB_t \\ &y(T) = \xi \end{aligned} \tag{15}$$

It is clearly that (15) is a backward stochastic control system in the form of system (5) when $(y,z,C)\in R^3$, then from theorem3.1, system (15) can be verified exact control easily, and optimum control is

$$C(t)=\frac{2a+b^2}{2\exp(\frac{b^2}{2}T+aT)-2}\exp(\frac{b}{2}B_t+\frac{a}{2}t)y_0 \tag{16}$$

where $y_0\in R$ is the initial assets and $a=\frac{2r\sigma^2+(r-\alpha)^2}{-\sigma^2}$, $b=\frac{2(r-\alpha)}{\sigma}$ .

**Example 4.2** Different from example 4.1, this time we assume that there is a bond and $m$ kinds of stocks in the stock market. The prices of the bond and each kind of stocks are defined as same as (10) and (11). The investment value at moment $t$ of the bond is also (12), but to the stocks

$$dS_i(t)=\alpha_i S_i(t)dt+\sigma_i S(t)_i dB_t \qquad i=1,2\cdots m \tag{17}$$

The total investment of investors at moment $t$ is $y(t)=S_0(t)+\sum_{i=1}^{m}\pi_i S_i(t)$, where $\pi_i(\sum_{i=1}^{m}\pi_i=1)$ is the ratio of assets invested in the $ith$ stock accounted for all assets invested stocks, then the total investments of investors at $t$ can be rewritten in form of

$$y(t)=\sum_{i=1}^{m}\pi_i[S_0(t)+S_i(t)]$$

Set $y_i(t)=\pi_i[S_0(t)+S_i(t)]$ we can get

$$\begin{aligned}&dy_i(t)=[ry_i(t)+(\alpha_i-r)\pi_i S_i(t)-\pi_i C(t)]dt+\pi_i\sigma_i S_i(t)dB_t \qquad i=1,2\cdots m\\&y_i(T)=\xi_i\end{aligned} \tag{19}$$

Let $z_i(t)=\pi_i\sigma_i S_i(t)$, $C_i(t)=\pi_i C(t)$ then (19) can be transformed

$$\begin{aligned}&-dy_i(t)=[-ry_i(t)+\frac{(r-\alpha_i)}{\sigma_i}z_i(t)-C_i(t)]dt-z_i(t)dB_t \qquad i=1,2\cdots m\\&y_i(T)=\xi_i\end{aligned} \tag{20}$$

From example 4.1, we know that (20) is exact control, and

$$C_i(t)=\frac{2a_i+b_i^2}{2\exp(\frac{b_i^2}{2}T+a_iT)-2}\exp(\frac{b_i}{2}B_t+\frac{a_i}{2}t)y_0 \qquad i=1,2\cdots m \tag{21}$$

where

$$a_i=\frac{2r\sigma_i^2+(r-\alpha_i)^2}{-\sigma_i^2}, \quad b_i=\frac{2(r-\alpha_i)}{\sigma_i}, \quad i=1,2\cdots m$$

Then the optimum consumption is

$$C(t)=\sum_{i=1}^{m}C_i(t) \qquad i=1,2\cdots m \tag{22}$$

## Conclusions

This paper gives a necessary condition of exact controllability for backward stochastic control systems, and three equivalent conditions of necessary and sufficient for the system to be exactly controllable when the system is linear. These conclusions are very valuable for research on the exact controllability of forward-backward stochastic control system.

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